Geometry of 3-manifolds in Euclidean space (Theory of singularities of smooth mappings and around it)

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By

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Abstract

This is a survey on the extrinsic geometry of 3-manifolds immersed in $\mathbb{R}^5$ and $\mathbb{R}^6$. We provide an advance of results describing the possible topological types of the curvature locus, together with a summary of the generic properties of the higher order singularities of height and distance squared functions published in previous papers.

§ 1. Introduction

The curvature ellipse at a point of a surface immersed in 4-space is a geometrical invariant that contains all the relevant information regarding the second order geometry of the surface at the considered point. This concept has been known since a long time ago [7] and it has proven to be a useful tool as illustrated in [8, 9, 4]. Its natural generalization to higher dimensional manifolds leads to the notion of curvature locus ([16]), which can be seen as the image of a convenient linear projection of the Veronese submanifold of order 2. For n-manifolds with $n > 2$, this set may present singularities that play a relevant role in the behaviour of the family of principal configurations on the submanifold. In this paper we focalize our attention in the case of 3-manifolds: We see that whereas the curvature locus of a 3-manifold immersed in $\mathbb{R}^5$ is a contractible

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subset of the normal plane at each point, the case of 3-manifolds in $\mathbb{R}^6$ happens to be particularly interesting for their curvature locus may exhibit different possible topological types which are conformal invariants of the 3-manifold. We describe here these possible topological types, providing examples that justify their possible realization as curvature loci. Our aim in this paper is to summarize some of the properties concerning the behaviour of the curvature locus providing an advance of results whose detailed proofs will appear in a a forthcomming paper [2]. We also include, for the sake of completeness, a summary of results concerning the generic behaviour of the sets of higher order singularities of height and distance squared functions on 3-manifolds immersed in Euclidean space that have been proven in [15].

We analyze in §2 the possible shapes of the curvature locus, describing its generic behaviour in the case of 3-manifolds in $\mathbb{R}^6$ and discussing some special cases. In §3 we analyze the generic contacts of 3-manifolds with hyperplanes and hyperspheres of the ambient space in terms of the singularities of the height and distance squared functions. This analysis gives rise to the concepts of binormal and asymptotic directions whose behaviour is tightly related to the relative position of the curvature locus at each point. We see that the local convexity of the submanifold can be charaterized in terms of the relative position of the curvature locus at a point and as a consequence we deduce that the local convexity is a sufficient condition for the existence of asymptotic directions. We describe some properties concerning subsets of points with geometrical interest, such as parabolic, inflection and semiumbilic points, ridges and flat ridges.

§ 2. Second order geometry of 3-manifolds in $\mathbb{R}^n$

Let M be a compact 3-manifold in $\mathbb{R}^{3+k}, k \geq 1$, locally given as the image of an embedding $f : U \to \mathbb{R}^{3+k}$ and denote by (x, y, z) the local coordinates on M. For each $p = f(x, y, z)$ we can take a basis of T_pM given by $B^f = \{f_x, f_y, f_z\}_p$, where $f_x = \frac{\partial f}{\partial x}$, etc. Let $\{e_1, e_2, \dots, e_k\}$ be an orthonormal normal frame on M, such that the orientation of the frame $\{f_x, f_y, f_z, e_1, \dots, e_k\}$ coincides with the orientation of the space $\mathbb{R}^{3+k}$.

Given a normal field ν on M, consider the linear map $d\nu : T_pM \to T_p\mathbb{R}^{3+k} = T_pM \oplus N_pM$ and let $\pi^T : T_pM \oplus N_pM \to T_pM$ be the tangential projection. The map $S_p^\nu = -\pi^T \cdot d\nu : T_pM \to T_pM$ is known as the *ν-shape operator* at p. The *second fundamental form of M along a normal vector field ν* is the bilinear map $II_p^\nu : T_pM \times T_pM \to \mathbb{R}$, defined by $II_p^\nu(\mathbf{v}, \mathbf{w}) = \langle \nu, d^2 f(\mathbf{v}, \mathbf{w})$. Observe that for each $p = f(u_0) \in M$ and $\mathbf{v} \in N_pM$, the coefficients of $II_p^{\mathbf{v}}$ do not depend on the choice of the vector field ν, $\nu(0) = \mathbf{v}$.

The second fundamental form $II_p : T_pM \times T_pM \to N_pM$ is the bilinear map given by $II_p(\mathbf{v}, \mathbf{w}) = \pi^\perp d^2 f(\mathbf{v}, \mathbf{w})$ that projects $d^2 f(\mathbf{v}, \mathbf{w})$ onto the normal space at p. This induces a linear map

$$A_p : N_pM \longrightarrow \mathcal{Q}_2$$

$$v \qquad \longmapsto II_p^v,$$

where $\mathcal{Q}_2$ is the 6-dimensional space of quadratic forms on T_pM. We say that a point $p \in M$ is of type M_i provided rank $A_p = i, i = 1, \cdots, 6$.

It is well known that S_p^ν is a self-adjoint operator. In fact, it satisfies

$$\langle S_p^\nu(v), w\rangle = II_p^\nu(v,w), \forall v, w \in T_pM.$$

The directions of the eigenvectors of S_p^ν are called *ν-principal directions* and its corresponding eigenvalues are the *ν-principal curvatures.* The points at which two of the ν-principal curvatures coincide are called *ν-quasiumbilic points.* This corresponds to having a tangent plane made of ν-principal directions. The points at which all the tangent directions are ν-principal directions are said to be *ν-umbilic points.* At these points, the 3 ν-principal curvatures coincide at p. The ν-quasiumbilic and ν-umbilic points form the critical set of the ν-principal configurations foliations on M.

A 3-manifold M is said to be *totally ν-umbilic* it all its points are ν-umbilic.

§ 2.1. The curvature locus

Definition 2.1. Given a point $p \in M \subset \mathbb{R}^{3+k}, k \geq 2$ and a unit vector $v \in S^2 \subset T_pM$, we denote by γ_v the *normal section* of M in the direction v, that is, $\gamma_v = M \cap \mathcal{H}_v$, where $\mathcal{H}_v = \{\lambda v\} \oplus N_pM$ is an affine subspace of codimension 2 through p in $\mathbb{R}^{3+k}$. The normal curvature vector $\eta(v)$ of γ_v at p belongs to N_pM. Varying v in $S^2 \subset T_pM$, we obtain a surface in the normal space N_pM. Since the normal curvature vector, $\eta(v)$ is the same at anthipodal points of $S^2 \subset T_pM$, we can view the image of η as the image of a projective plane P^2 in $N_pM \equiv \mathbb{R}^k$. We call it the *curvature locus* of M at p.

We can also view the curvaure locus as the image of the unit tangent sphere at $p \in M$ via $II(v,v)$. By taking spherical coordinates, we can write [1]:

$$\eta : S^2 \subset T_pM \longrightarrow N_pM$$

$$(\theta, \phi) \qquad \longmapsto \eta(\theta, \phi),$$

with

$$\begin{aligned}\eta(\theta,\phi) = H \ + \ \big(1 + 3\cos(2\phi)\big)B_1 \ + \ \cos(2\theta)\big(\sin(\phi)\big)^2 B_2 \\ + \sin(2\theta)\big(\sin(\phi)\big)^2 B_3 \ + \ \cos(\theta)\sin(2\phi)B_4 \ + \ \sin(\theta)\sin(2\phi)B_5,\end{aligned}$$

where

$$H = \tfrac{1}{3}\big(f_{xx} + f_{yy} + f_{zz}\big), \ B_1 = \tfrac{1}{12}\Big(-f_{xx} - f_{yy} + 2f_{zz}\Big),$$

$B_2 = \frac{1}{2}\Big(f_{xx} - f_{yy}\Big)$, $B_3 = f_{xy}$, $B_4 = f_{xz}$ and $B_5 = f_{yz}$.

The value of the normal field H at $p \in M$, is classically known as the *mean curvature vector* of M at p.

The *first normal space* of M a p is the subspace of N_pM spanned by the vectors $\{f_{xx},\ f_{xy},\ f_{xz},\ f_{yy},\ f_{yz},\ f_{zz}\}$. Therefore, we can write

$$N_p^1 M \;=\; \big\langle H,\ B_1,\ B_2,\ B_3,\ B_4,\ B_5\big\rangle_{(p)}.$$

We denote by Aff_p the affine hull of the curvature locus in N_pM and by E_p the linear subspace of N_p^1M parallel to Aff_p. So $E_p = \big\langle B_1,\ B_2,\ B_3,\ B_4,\ B_5\big\rangle_{(p)}$ and we have that $p \notin Aff_p$ if and only if $H \notin E_p$ (or equivalently, if H is transversal to Aff_p). It follows that the curvature locus is contained is an affine subspace of N_pM with dimension ≤ 5.

§ 2.2. Possible topological types for the curvature locus

The classical Veronese surface of order 2 is a surface given by the image of the map:

$$\xi(u,v,w) = \Big(u^2, v^2, w^2, \sqrt{2}uv, \sqrt{2}uw, \sqrt{2}vw\Big), \quad \forall (u,v,w) \in S^2.$$

This surface is substantially contained in a 4-sphere (and hence in a hyperplane) of $\mathbb{R}^6$

Proposition 2.2 ([1]). *Given a 3-manifold M in $\mathbb{R}^{3+k}, k \geq 2$, and $p \in M$, the curvature locus at p is isomorphic (and thus diffeomorphic) to the classical Veronese surface of order 2 composed with an affine map from $\mathbb{R}^5$ to $N_pM \equiv \mathbb{R}^k$. The rank of the associated linear transformation coincides with the dimension of the subspace $Aff_p \subseteq N_pM$ (note that the curvature locus is a surface substantially immersed in Aff_p). Moreover,*

a) $p \in M_6$ if and only if the curvature locus at p is difeomorphic to the Veronese surface which is substantial in the 5-space Aff_p and $H(p) \notin E_p$.

b) $p \in M_k$, for $k = 3, 4, 5$, if and only if either $H(p) \notin E_p$ and dim $Aff_p = k-1$, or $E_p = Aff_p$ and dim $Aff_p = k$.

c) $p \in M_2$ if and only if the curvature locus at p is either a non radial segment non parallel to H (i.e. p is a semiumbilic point), or a closed region in the plane $E_p = Aff_p$.

d) $p \in M_1$ if and only if the curvature locus at p is either a radial segment (flat semiumbilic point), or a point distinct from p (umbilic point).

e) $p \in M_0$ if and only if the curvature locus at p coincides with p (flat umbilic point).

Since we are mainly interested in the study of 3-manifolds immersed in $\mathbb{R}^5$ and $\mathbb{R}^6$ we shall concentrate our atention in the case $p \in M_k, k \leq 3$.

The projection of a Veronese surface in a 3-dimensional space is known as a *Steiner surface.* The possible topological type of Steiner surfaces in the real space have been studied in [3]. By analyzing the different possibilities and taking into account that the curvature locus is a compact surface given by the image of a projective plane, whose implicit expression is given by a polynomial of degree 2 or 4, we arrive to the following:

Theorem 2.3. *([1],[2]) The curvature locus at a point p of type M_3 in a 3-manifold M immersed into $\mathbb{R}^6$ is isomorphic to one of the following:*

1) A Roman Steiner surface, a Cross-Cap surface, a Steiner surface of type 5, a Cross-Cup surface, an ellipsoid, or a (compact) cone, provided the mean curvature vector $H(p)$ lies in E_p.

2) An elliptic region, a triangle, a (compact) planar cone, or a planar projection (of type 1, 2 or 3) of the Veronese surface, provided the mean curvature vector $H(p)$ does not lie in E_p.

Geometrical description of the different models:

1) The *Steiner surface of type 1 or Roman Steiner surface* is given in implicit form by

$$x^2y^2 + x^2z^2 + y^2z^2 - xyz = 0.$$

This surface has 6 pinch points lying at the ends of 3 double curves with mutual intersection at a triple point.

2) The *Steiner surface of type 3 or Cross-Cap surface* is given in implicit form by

$$4x^2(x^2 + y^2 + z^2 + z) + y^2(y^2 + z^2 - 1) = 0.$$

This surface has two pinch points and no triple points.

3) The *Steiner surface of type 5*, given in implicit form by

$$x^2(z-1)^2 + y^2(y^2 + z^2 - 1) = 0,$$

has two pinch points, one of them being a triple point too.

4) The *Steiner surface of type 6 or Cross-Cup surface* has one triple point and no singular points. Its expression in the implicit form is:

$$-\frac{5}{4}x^4 + 3x^3y - \frac{5}{2}x^2y^2 + xy^3 - \frac{1}{4}y^4 - 3x^2z^2 + 4xyz^2 - y^2z^2 - z^4 + \frac{7}{2}x^3 - \frac{11}{2}x^2y + \frac{5}{2}xy^2 - \frac{1}{2}y^3 + 5xz^2 - 3yz^2 - \frac{13}{4}x^4 + \frac{5}{2}xy - \frac{1}{4}y^2 - 2z^2 + x = 0.$$

Remark. We observe that Steiner surfaces of types 2 and 4 also admit degree 4 implicit equations and can be respectively transformed into surfaces of Types 1 or 3, through convenient linear complex transformations [3]. Nevertheless, when considered as real polynomials they lead to non compact algebraic surfaces. On the other hand, the Boy surface, which provides an immersion of projective plane in $\mathbb{R}^3$, which is homeomorphic to the Roman Steiner surface, is an algebraic variety of degree 6 and cannot be realized as a curvature locus.

The planar regions are displayed in Figures 1 to 6.

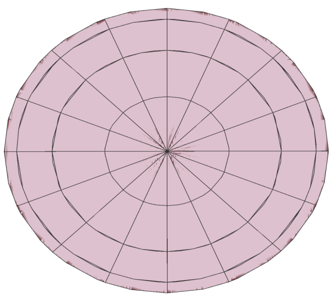

Figure 1. Eliptical region

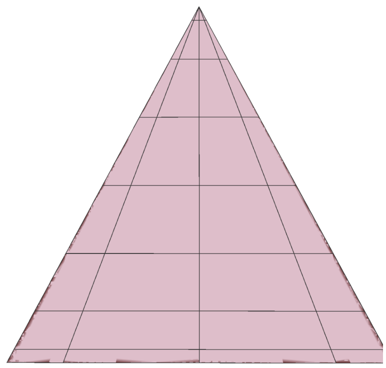

Figure 2. Triangular region

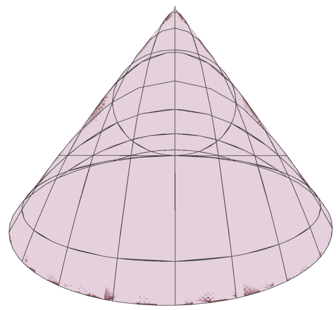

Figure 3. Planar cone

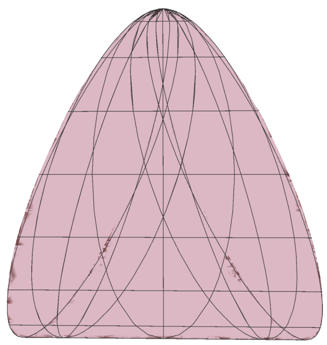

Figure 4. Planar type 1

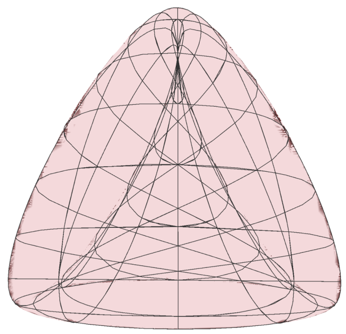

Figure 5. Planar type 2

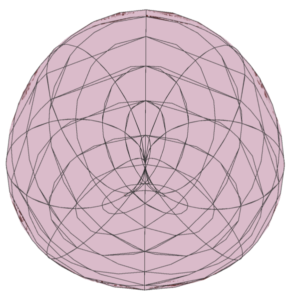

Figure 6. Planar type 3

Theorem 2.4. *The curvature locus at a point p of type M_2 in a 3-manifold M immersed into $\mathbb{R}^6$ (or $\mathbb{R}^5$) may be isomorphic to one of the following: elliptic region, triangle, (compact) planar cone, planar projections (of type 1, 2 or 3) of the Veronese surface, or to a non radial segment.*

Proposition 2.5 ([2])). *A generic immersion of a 3-manifold M into $\mathbb{R}^6$ is exclusively made of M_3 points. Moreover, the curvature locus at each point $p \in M$ is a substantial surface in the 3-dimensional space N_pM on an open and dense subset of M. The complement of this subset is made of M_3-points of for which the curvature locus is one of the plane regions described in theorem 2.3.*

An interesting particular case is given by the 3-manifolds with flat normal bundle in $\mathbb{R}^n, n \geq 5$. We say that a 3-manifold M has *flat normal bundle* it it admits a globally defined parallel normal frame. This is equivalent to asking that M has *vanishing normal curvature* ([17]). But this amounts to say that M has a unique normal configuration, that is, there are tangent vectors v_1, v_2, v_3 at each $p \in M$ which are ν-principal directions for any normal field ν on M at p. The next result, proven in [16], provides a characterization of this property for 3-manifolds in $\mathbb{R}^5$ in terms of the shape of curvature locus. We observe that this can be easily extended to higher codimension.

Theorem 2.6 ([16]). *Given a 3-manifold in $\mathbb{R}^5$, we have that the curvature locus of M at p is a triangle, given by the convex hull of the points $\eta(X_i) \in N_pM$, if and only if M admits a unique principal configuration (i.e., M has vanishing normal curvature) and $\{X_i\}_{i=1}^3$ are the (univocally defined) principal directions at p.*

Example 2.7. The following examples show that all the topological types of curvature locus corresponding to the non planar models can be realized. The figures have been obtained with the aid of the program "ImmersionR3R6" due to A. Montesinos Amilibia [13].

1) The immersion $f : \mathbb{R}^3 \to \mathbb{R}^6; f(x, y, z) = (x, y, z, x^2 + y^2 - z^3, xy, xz)$ defines a 3-manifold in $\mathbb{R}^6$ whose curvature loci at points near to the origin are isomorphic to the Cross-Cap surface. The Figure 7 shows the curvature locus at the point $((0.0126, -0.2652, 0)$.

2) The curvature locus of immersion $f : \mathbb{R}^3 \to \mathbb{R}^6; f(x, y, z) = (x, y, z, xy, xz, yz)$ at points near to the origin is isomorphic to the roman Steiner surface as illustrated by Figure 8

3) The curvature locus of the the immersion $f : \mathbb{R}^3 \to \mathbb{R}^6; f(x, y, z) = (x, y, z, 4x^2 + \frac{1}{2}z^2 + 2xz, 2z^2, 2yz)$ at points near the origin is a projection of the Veronese surface given by the Cross-Cup surface as shown in figure 9.

4) Figure 10 shows a curvature locus, isomorphic to a Steiner surface of type 5, corresponding to the immersion $f : \mathbb{R}^3 \to \mathbb{R}^6; f(x, y, z) = (x, y, z, 2xy, x^2 + y^2 + 3z^2, 2xz)$ at the origin.

5) Figure 11 shows the curvature locus of the immersion $f : \mathbb{R}^3 \to \mathbb{R}^6; f(x, y, z) = (x, y, z, x^3 + y^3, y^2z, z^3)$ at the point $((0.1, -0.2, 0)$ which is given by a cone.

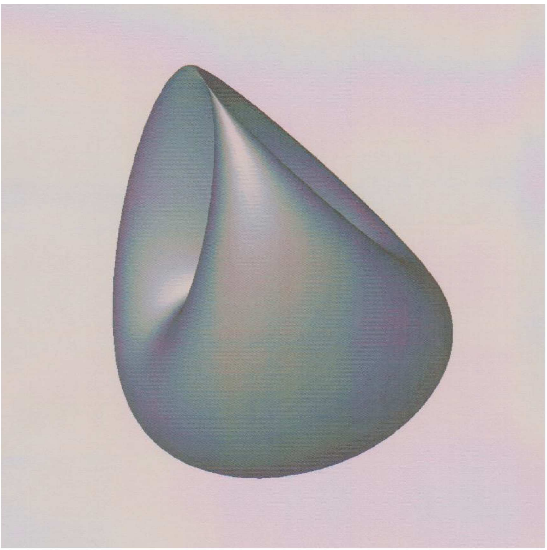

Figure 7. Cross-Cap surface.

5) The curvature locus of the immersion $f : \mathbb{R}^3 \to \mathbb{R}^6$; $f(x, y, z) = (x, y, z, x^2+y^2-z^2, x^2+2xy+y^2+\frac{3}{2}w^2, x^2+y^2+z^2+2yz)$ at the point $((0, 0, 0)$ is an ellipsoid. We observe that this is a very "unstable" situation. As near as we want to the origin, the curvature locus presents 2 cross-cap points.

Example 2.8. We provide now some examples of (local) immersions of 3-manifolds into $\mathbb{R}^6$ having a point of type M_3 at the origin and whose curvature locus is one of the planar models. These examples illustrate the possibility of realizing all the above planar models as the curvature locus of some immersion.

1) The curvature locus at the origin of the immersion $f(x, y, z) = (x, y, z, x^2+z^2, xy, x^2+y^2+z^2)$ is an elliptic region.

2) The curvature locus at the origin of the immersion $f(x, y, z) = (x, y, z, x^2, y^2, z^2)$ is a triangular region.

3) The curvature locus at the origin of the immersion $f(x, y, z) = (x, y, z, -2y^2-z^2, xz, x^2+y^2+z^2)$ is a planar cone.

4) The curvature locus at the origin of the immersion $f(x, y, z) = (x, y, z, 2xy+2yz, 2z^2, x^2+y^2+z^2)$ is a planar region of type 1.

5) The curvature locus at the origin of the immersion $f(x, y, z) = (x, y, z, 2x^2+z^2+xy, xz, x^2+y^2+z^2)$ is a planar region of type 2.

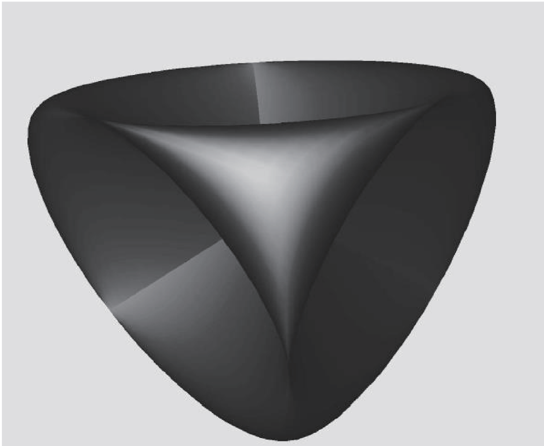

Figure 8. Roman Steiner surface

6) The curvature locus at the origin of the immersion $f(x,y,z) = (x,y,z,xy,yz,x^2 + y^2 + z^2)$ is a planar region of type 3.

§ 3. Higher order geometry

§ 3.1. Height functions, binormals and asymptotic directions

Suppose that M is given as the image of an embedding $f : M \to \mathbb{R}^{3+k}$ and consider the family of height functions on M,

$$\begin{aligned} \lambda(f) : M \times S^{2+k} &\longrightarrow \mathbb{R} \\ (x,v) &\longmapsto \langle f(x), v\rangle = f_v(x). \end{aligned}$$

The singularities of the functions f_v describe the contacts of M with the hyperplanes of $\mathbb{R}^{3+k}$. We observe that a height function f_v has a singularity at $x \in M$ if and only if v is normal to M at x and we have that the singularity type of f_v at x determines the contact of M with the hyperplane orthogonal to v passing through x (see [12]).

It follows from Montaldi's genericity theorem [12] that there is a residual subset $\mathcal{E}$ of embeddings of $\mathbb{R}^3$ into $\mathbb{R}^{3+k}$ with the Whitney C^∞-topology, such that for any $f \in \mathcal{E}$, the familiy $\lambda(f)$ is a generic family of functions on $\mathbb{R}^{3+k}$. Moreover we have that for $k \leq 3$, given $f \in \mathcal{E}$, the germ of $\lambda(f)$ at any point (x,v) is a versal unfolding of the germ of f_v at x and thus the functions f_v may only have singularities of codimension less or equal to $k+2$. In particular, those of corank one belong to the series $\{A_j\}_{j\geq 1}$ and have $\mathcal{A}$-codimension $j-1$.

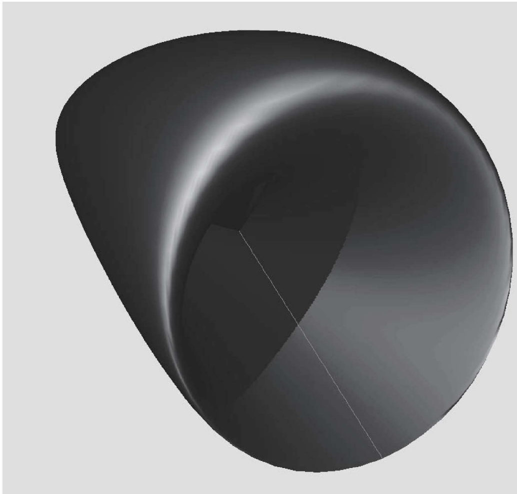

Figure 9. Cross-Cup surface.

Definition 3.1. A unit vector $v \in N_xM$ is said to be a *binormal* provided that x is a non Morse singularity of f_v, that is, a singularity of $\mathcal{A}$-codimension at least 1. In such case, we have that Ker $Hess(f_v)(x) \neq 0$ and any direction $u \in Ker(Hess(f_v)(x))$ is said to be an *asymptotic direction associated to v.*

Definition 3.2. A unit vector $v \in N_xM$ for which f_v has a singularity of type $A_{j\geq k}$ is said to be a *strong binormal direction* of M at x. The direction determined by the kernel of $\mathrm{Hess}(f_v)(x)$ is said to be a *strong asymptotic direction.*

The number of strong binormal directions at each point of M is finite and may vary from a point to another. A generic 3-manifold can be subdivided into open regions with different constant numbers of strong binormal directions which are separated by a discriminant surface.

The behaviour of the generic singularities of height functions on 3-manifolds in $\mathbb{R}^4$ and related properties has been studied by A. C. Nabarro in [14]. The case of 3-manifolds generically immersed in $\mathbb{R}^n, n > 4$ has been considered in [15], where the following property was shown.

Proposition 3.3. *([15]) The set of flat ridges of a 3-manifold generically immersed in $\mathbb{R}^{3+k}, k \geq 2$ is a surface with normal crossings and possible isolated cross-caps.*

Following the criteria established in [9] for surfaces in $\mathbb{R}^4$, a straightforward generalization of the concepts of parabolic and inflection point for a 3-manifold in $\mathbb{R}^6$ would be the following:

Figure 10. Steiner surface of type 5

Definition 3.4. A point p of a 3-manifold M immersed in $\mathbb{R}^6$ is said to be a *parabolic* point if the curvature locus of M at p passes through p.

Definition 3.5. A point p of a 3-manifold M immersed in $\mathbb{R}^6$ is said to be an *inflection point* if there exists a unit normal vector $v \in N_pM$ such that the height function f_v has a singularity of corank 3 at p

Lemma 3.6. *The inflection points of a 3-manifold immersed in $\mathbb{R}^6$ are the points of type $M_i, i \leq 2$. These points do not appear generically on 3-manifolds immersed in $\mathbb{R}^6$.*

The following approach, proposed by D. Dreibelbis in [6], leads to an alternative generalization based in the idea of conjugate directions with respect to a quadratic form: A pair (u, w) of vectors in T_pM is said to be a *conjugate pair* if $II_v(u, w) = 0$, $\forall v \in N_pM$ (i.e., u and w are conjugate for all second fundamental forms defined by all normal vectors to M at p). Then a vector u is said to be a *conjugate vector* at $p \in M$ if there exists $w \in T_pM$ such that (u, w) is a conjugate pair. In case that $u = w$ we say that u is a *self-conjugate vector*. In this case it is shown that u *is a conjugate vector at* $p \in M$ *if and only if it is an asymptotic vector of* M *at* p. This viewpoint leads to the following alternative concepts of parabolic and inflection points for 3-manifolds in $\mathbb{R}^6$:

Definition 3.7. A point p is said to be *parabolic* if it has exactly one self-conjugate vector (including multiplicity).

A point p is an *inflection point* if it has more than one (counting multiplicity and possibly complex) self-conjugate vectors.

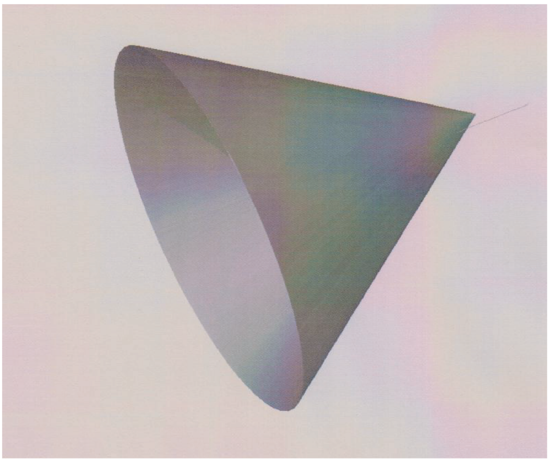

Figure 11. Curvature locus given by a cone.

We observe that whereas both definitions of parabolic point are equivalent, the second generalization of inflection points, to which we shall refer here as *multiple-parabolic points*, differs from the first one but it happens to be more useful in the study of 3-manifolds immersed in $\mathbb{R}^6$. In fact, 3-manifolds in $\mathbb{R}^6$ present generically multiple-parabolic points along codimension 2 subsets. The study of the possible algebraic structures of the asymptotic vectors at parabolic and multiple-parabolic points, as well as the description of the generic behaviour of the parabolic set of a 3-manifold M has been carried out in [6]. It is there shown that generically the parabolic set is a surface with normal crossings and isolated cross-cap points. The multiple-parabolic points (or inflection points in Dreibelbis' sense) occur at the self-intersections and cross-cap points of the parabolic set. In particular, it is observed in [6] that *the structure of the parabolic set at a point p reflects the behaviour of the curvature locus at p, in the sense that p is a cross-cap (resp. a triple, a double or a regular) point of the parabolic set in M if and only if the origin p is a cross-cap (resp. a triple, a double or a regular) point of the curvature locus in N_pM.*

A further consequence of the analysis made in [6] is the following characterization of the cone of binormal directions in terms of the curvature locus.

Proposition 3.8. *The cone of binormal directions is made of all the orthogonal directions to the cone subtended by the curvature locus from the origin p of N_pM together with the orthogonal directions to those determined by the singular points of the curvature locus in N_pM.*

Definition 3.9. A tangent hyperplane Π is said to be a *locally support* hyperplane for the submanifold M at the point p if M is locally contained at p in one of the two closed half-spaces determined by Π in $\mathbb{R}^{3+k}$. We say that M is *locally convex* at $p \in M$ if there is a locally support hyperplane Π of $\mathbb{R}^{3+k}$ at p. If this plane has a non-degenerate contact (i.e. of Morse type) with M, then we say that M is *strictly locally convex* at p.

We can express this in terms of height functions as follows: The 3-manifold M is *locally convex* at $p \in M$ if there is a height function f_v on M such that p is a local maximum (or minimum) of f_v. Then we can say that M is *strictly locally convex* at p provided p is a Morse singularity of f_v.

We have the following geometric characterization of local convexity in terms of the curvature locus.

Theorem 3.10. *A 3-manifold $M \subset \mathbb{R}^{3+k}$ is strictly locally convex at p if and only if the origin of the normal plane (identified with $p \in \mathbb{R}^{3+k}$) is not contained inside the convex hull of the locus of curvature of M at p.*

This together with Proposition 3.8 leads to the following:

Corollary 3.11. *The local convexity at a point $p \in M$ implies the existence of degenerate normal directions and hence the existence of asymptotic directions at p.*

In the particular case of a 3-manifold M in $\mathbb{R}^5$, it was shown in [10] that there is at least one binormal direction at each point of M. On the other hand, it has been proven in [16] that the local convexity implies the existence of exactly 3 binormal directions (counting their multiplicities) on 3-manifolds in $\mathbb{R}^5$.

§ 3.2. Distance squared functions family, focal sets and semiumbilics

The *family of distance squared functions* over a 3-manifold M immersed in $\mathbb{R}^{3+k}$ is defined to be

$$\begin{aligned} d_f : M \times \mathbb{R}^{3+k} &\longrightarrow \mathbb{R} \\ (x, a) \qquad &\longmapsto d_a(x) = \|f(x) - a\|^2. \end{aligned}$$

This family measures the contacts of M with the hyperspheres of $\mathbb{R}^{3+k}$. A point $x \in M$ is a singular point of a function d_a if and only if the vector $a - f(x)$ lies in the normal subspace N_xM of M at x. The singularity type of d_a at x determines the contact of M with the hypersphere with center a passing through x.

Again, it follows from Montaldi's genericity theorem [11, 12] that there is a residual subset $\mathcal{E}'$ of embeddings of $\mathbb{R}^3$ into $\mathbb{R}^{3+k}$ with the Whitney C^∞-topology such that for any $f \in \mathcal{E}'$, the familiy d_f is a generic family of functions on $\mathbb{R}^{3+k}$. The generic singularities of the family d_f were first studied by Porteous [18], who observed that

$$\Sigma(d) = \{(f(x), a) \in M \times I\!R^{3+k} | \frac{\partial d_a}{\partial x} = 0\} = NM.$$

The restriction of the projection $\pi : M \times \mathbb{R}^{3+k} \to \mathbb{R}^{3+k}$ to $\Sigma(d) = NM \subset M \times \mathbb{R}^{3+k}$ (= *catastrophe map* associated to the family d) is the normal exponential map exp_N of M.

Definition 3.12. The bifurcation set of the family d_f,

$$\mathcal{B}(d) = \{a \in \mathbb{R}^{3+k} | \exists x \in M \text{ where } d_a \text{ has a degenerate singularity}\},$$

is the *focal set* of M. The hyperspheres tangent to M whose centers lie in $\mathcal{B}(d)$ are called *focal hyperspheres* of M. The centers of the focal hyperspheres at $x \in M$ define distance-squared functions with a singularity of type A_3 or worse.

Definition 3.13. The *ridges* of M are the singularities of type A_i, $i \geq k+2$, for the different distance-squared function on M. The *highest-order ridge points* are the singularities of type A_i, $i \geq 4+k$, for some distance-squared function. These are, generically, isolated points.

Proposition 3.14. *([15]) The set of ridges on a generic* 3*-manifolds in* $\mathbb{R}^{3+k}, k \geq 2$ *is a surface with normal crossings and possible isolated cross-caps.*

Definition 3.15. The corank 3 singularities of distance squared functions on a 3-manifold M immersed in $\mathbb{R}^{3+k}, k = 2, 3$ are called *semiumbilic* points of M.

Proposition 3.16. *([2]) We have the following characterization of semiumbilics in terms of the curvature locus:*

a) For a 3*-manifold* M *immersed in* $\mathbb{R}^6$ *the semiumbilic points are the points at which the curvature locus degenerates into a planar region (including a segment, or a point as particular cases).*

b) A point p *of a* 3*-manifold* M *immersed in* $\mathbb{R}^5$ *is semiumbilic if and only if the curvature locus at* p *is a segment (or, as a special case, a point).*

Remark. We can view the inflection points as a special case of semiumbilic points at which the mean curvature vector H_p lies in the vector subspace E_p spanned by the curvature locus at p.

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